Topics on the essential self-adjointness for Klein-Gordon type operators on spacetimes

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In this talk we discuss the essential self-adjointness of Klein-Gordon type operators (or wave operators) on curved spacetimes. These results are based on joint works with Kouichi Taira (Ritsumeikan University).

We consider 3 types of spacetimes. The first one is the asymptotically flat spacetime. Typically, we consider operators of the form:

$$P = \sum_{j,k=0}^{n} D_j g^{jk}(x) D_k + \frac{1}{2}\sum_{j=0}^{n}(D_j u_j(x) + u_j(x) D_j) + v(x),$$

on $L^2(\mathbb{R}^{n+1})$, where $D_j = -i\frac{\partial}{\partial x_j}$, $j = 0, 1, 2, \dots, n$, $n \geq 1$. This corresponds to the wave operator on a spacetime with the pseudo-metric $\{g_{jk}\} = \{g^{jk}\}^{-1}$ on $\mathbb{R}^{n+1}$. We always suppose $\{g^{jk}(x)\}$ is invertible, but not necessarily positive. We say the metric is *asymptotically flat*, if $\{g^{ij}(x)\}$ converges to a flat metric $\{g_0^{jk}\}$. In particular, if it converges to the Minkowski metric: $g_0^{ij} = \epsilon_i \delta_{ij}$, where $\epsilon = (1, -1, \cdots, -1)$, then it is called *asymptotically Minkowski*. If the perturbations: $g^{ij}(x) - g_0^{jk}$, $u_j(x)$ and $v(x)$ satisfy long-range type decay conditions as $|x| \to \infty$, the essential self-adjointness was proved by Vasy (2020, JST) and Nakamura-Taira (2021, AHL), and a simplified proof is obtained recently by Nakamura-Taira (preprint 2022).

The second model is the asymptotically static spacetime. We consider operators of the same form but on $X = \mathbb{R} \times M$, where M is a closed Riemannian manifold. Here $x_0 = t$ is the variable in $\mathbb{R}$, and $(x_1, \dots, x_n)$ corresponds to the (local) coordinates in M. Let $\{q_{jk}\}$ be the metric on M. If g converges to $dt^2 - q^{jk}$ as $|t| \to \infty$, then the spacetime (corresponding to g) is called *asymptotically static*. The essential self-adjointness of wave operators on asymptotically static spacetimes are proved recently by Nakamura-Taira (2022, CMP) under short-range type conditions.

The third model is the expanding spacetime. We consider the same manifold as in the asymptotically static case, but we suppose the metric is of the form

$$g(t, x) \sim dt^2 - |t|^{2\alpha} q(x) \quad \text{as } |t| \to \infty$$

with $\alpha > 0$. This model corresponds to a spacetime expanding of the order $O(|t|^\alpha)$ as $|t| \to \infty$. We can prove the essential self-adjointness of the wave operaor on this space under certain conditions on the perturbations, provided $0 < \alpha < 1$. This part is a work in progress.

We discuss these results and sketch main ideas of the proof, with emphasis on the relationship to the geometry of the null geodesics.